APPLICATION OF THE HYDRODYNAMIC ANALOGY TO THE FLOW IN A SUPERSONIC NOZZLE

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We examine the flow of a supersonic stream in an annular duct. In approximation of the hydrodynamic analogy we have given consideration to both friction and heat transfer. A general equation is presented, as are the solutions for several special cases.

The problem of the flow of a viscous gas with heat transfer in a duct of variable cross section (involving the use of a basic relationship $\zeta = 8\mathrm{Sn}$ from the hydrodynamic theory of heat transfer) has been solved in quadratures in references [1–3].

It has been demonstrated experimentally in reference [4] that the above-indicated relationship between the intensity of heat transfer and the hydrodynamic resistance is disrupted on transition through the speed of sound. However, beginning from Mach number values of $M \approx 1.48$, we can regard the use of this relationship as permissible within the accuracy of engineering practice.

The unique feature of the solution obtained in reference [3] lies in the fact that it reduces to the determination of the function $\varphi(M)$, characterizing the deviation of the flow from the ideal.

Here, as in [3], it is assumed that the sum $|k_1| + |k_2|$ of the absolute values of the slopes for the generatrices of the inside and outside duct surfaces is fairly small. The stagnation-temperature recovery factor in the boundary layer is assumed to be equal to 1.

From the equation of motion

$$\frac{1}{T}\frac{d(M^2T)}{d\xi} = -\frac{1}{\gamma p^2}\frac{dp^2}{d\xi} - \frac{\zeta}{2}M^2\frac{1}{1+K\xi}, \quad (1)$$

from the continuity equation

$$\frac{M^2p^2}{T}(1+K\xi)^2(Z+L\xi)^2 = \mathrm{const} \quad (2)$$

and from the energy equation (using the relationship $\zeta = 8\mathrm{Sn}$), we obtain

$$\frac{(M^2-1)\,dM^2}{M^2\left(1+\frac{\gamma-1}{2}M^2\right)} = \left[1+\frac{\zeta}{8K}(1-\theta)-\frac{\zeta}{8K}(1+\theta)\gamma M^2\right]\times$$

$$\times\, d\ln(1+K\xi)^2 + d\ln(Z+L\xi)^2. \quad (3)$$

The quantity θ can be regarded as approximately constant, if the wall temperature is constant, and if the stagnation temperature changes only slightly [3]. It is not difficult to demonstrate that Eq. (3) reduces to an Able equation of the first kind [5].

Because the solution of this equation is cumbersome and, consequently, because of the unavoidable difficulties in the practical application of this solution, let us dwell on several special cases.

1. An annular duct with a constant cross-sectional area, i.e., $k_1 = k_2 = 0$; $y_{01} \neq 0$.

Equation (3) assumes the form

$$\frac{(M^2-1)\,dM^2}{M^2\left(1+\frac{\gamma-1}{2}M^2\right)} = \frac{\zeta}{4}[1-\theta-(1+\theta)\gamma M^2]\,d\xi. \quad (4)$$

The solution of this equation follows:

$$\varphi_1^n\varphi_2^m\left(\frac{M_0^2}{M^2}\right)^m = \exp\left(\frac{\xi\zeta}{4}\right),$$

where

$$\varphi_1 = \frac{\left(1+\frac{\gamma-1}{2}M^2\right)[1-\theta-(1+\theta)\gamma M_0^2]}{\left(1+\frac{\gamma-1}{2}M_0^2\right)[1-\theta-(1+\theta)\gamma M^2]};$$

$$\varphi_2 = \frac{(1+\theta)\gamma M^4 - \left[(1-\theta)-\frac{2\gamma(1+\theta)}{\gamma-1}\right]M^2 - \frac{2(1-\theta)}{\gamma-1}}{\gamma(1+\theta)M_0^4 - \left[1-\theta-\frac{2\gamma(1+\theta)}{\gamma-1}\right]M_0^2 - \frac{2(1-\theta)}{\gamma-1}};$$

$$n = \frac{3(1-\theta)-\gamma(1+3\theta)}{2(\gamma-1)+\frac{\gamma(1+\theta)}{1-\theta}}; \qquad m = \frac{1}{2(1-\theta)}.$$

When $y_{01} = 0$ and $\theta = 1$, we obtain a special case of flow in an insulated tube of constant cross section [6].

2. An annular duct of variable cross section with generatrix slopes equal in various directions: $k_1 < 0$; $k_2 > 0$; $|k_1| = |k_2| = k$. In this case $K = 2k$ and $L = 0$ and the solution to Eq. (3) has the form

$$\frac{q(M)}{q(M_0)}\varphi(M) = \frac{1}{1+2k\xi}.$$

Here $q(M)$ is the reference density of the mass flow, and the value of $\varphi(M)$ is the same as in [3].

3. An axisymmetric nozzle without a centerbody [3]. In this case, $k_1 = 0$; $y_{01} = 0$.

4. A plane nozzle of infinite width. The height of the inlet section is denoted by y_{01}, and $\xi = x/y_0$. In the same way as above, we can derive the equation

$$\frac{(M^2-1)\,dM^2}{M^2\left(1+\frac{\gamma-1}{2}M^2\right)} =$$

$$= \left[1 + \frac{\zeta}{8K}(1-\theta) - \frac{\zeta}{8K}(1+\theta)\gamma M^2\right] d\ln(1+K\xi)^2,$$

whose solution is

$$\frac{q(M)}{q(M_0)}\varphi(M) = \frac{1}{1+K\xi},$$

where $\varphi(M)$ has the same form as in [3], with the exception of $k = K/2$.

By calculating the Mach number M at any cross section of the nozzle, we can determine all of the remaining flow parameters in a manner analogous with that of [3].

NOTATION

M is the Mach number; T is the temperature; T_1 is the wall temperature; ζ is the friction factor; Sn is the Stanton number; q(M) is the reduced density of mass flow; k_1 and k_2 are the slopes of the inner and outer surfaces; y_{01} and y_{02} are the radii in the inlet section; x is the abscissa directed along the nozzle axis;

$$Y_0 = y_{02} - y_{01};\ K = k_2 - k_1;\ L = k_2 + k_1;\ \xi = \frac{x}{Y_0};$$

$$Z = 1 + \frac{2y_0}{Y_0};\ \theta = \frac{T_1}{T\left(1+\frac{\gamma-1}{2}M^2\right)}.$$

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